

NUCLEATION OF CRACKS IN A PERFORATED FUEL CELL

V. M. Mirsalimov

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A mathematical model is constructed for crack nucleation in an isotropic fuel cell (heat-releasing solid material) attenuated by a biperiodic system of cooling cylindrical channels with a circular cross section. Cracks are assumed to appear with increasing heat-release intensity in the bulk of the material. The solution of the problem on equilibrium of an isotropic perforated fuel cell with crack nuclei reduces to the solution of a nonlinear singular integral equation with a Cauchy-type kernel. The solution of the latter equation yields the forces in the band of crack nucleation. The condition of crack nucleation is formulated with allowance for the criterion of ultimate extension of bonds in the material.

Key words: *perforated heat-releasing material, pre-fracture zone, bonds between crack faces.*

Formulation of the Problem. Design of certain types of fuel cells in nuclear reactors requires calculating temperature stresses in a continuous medium with cylindrical channels aligned in parallel. The problem of thermoelastic equilibrium of a plane attenuated by a biperiodic system of holes was considered in [1–3]. As the heat-release intensity q increases, zones of elevated stresses are formed in such a material around the holes; these zones are arranged biperiodically. Surface cracks can appear in zones with elevated stresses. The problem of crack nucleation is an important problem of fracture mechanics. Formulation of this problem substantially expands the original Griffith concept, which implies that there are always many tiny cracks in any material. Formation (nucleation) of a crack under loading is consistent with fractographic observations. As the heat-release intensity increases, there arise pre-fracture zones on the hole surface, which are modeled as regions with weakened interparticle bonds in the material. Interaction of the faces of these zones is modeled by introducing a band of pre-fracture of bonds between these faces with a prescribed deformation diagram. The physical nature of these bonds and the sizes of the pre-fracture zones depend on the type of the material. As these zones (interlayers in the material) are small, as compared with the remaining part of the configuration, they may be mentally removed and replaced by cuts whose surfaces interact with each other according to a certain law corresponding to the action of the removed material. Allowance for these effects in problems of fracture mechanics is an important but also a difficult problem.

In the case considered, crack nucleation in the material is the process of the transition of the pre-fracture zone to the zone with broken bonds between the material surfaces. The size of the pre-fracture zone is unknown in advance and has to be determined during solving the problem.

The studies of the emergence of zones with a distorted structure of the material show that the initial stage of the pre-fracture zone is a narrow extended layer; as the load increases, a secondary system of zones containing the material with partly violated bonds suddenly appears [4–8].

We use the following assumptions: 1) the heat-release intensity is uniform over the entire bulk of the material, and the material can freely expand in all directions; 2) heat removal occurs through the channel surfaces only; 3) the maximum temperature difference in the medium is small; hence, the properties of the material in this range of temperatures remain unchanged; 4) the material is not affected by external forces; it experiences only the action of internal thermal stresses; 5) the material is in the steady state.

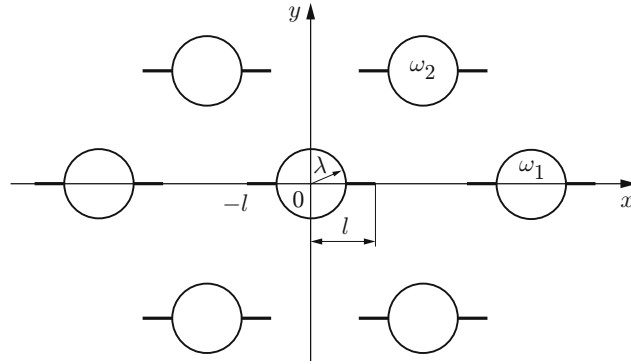


Fig. 1. Schematic of the problem of crack nucleation in a heat-releasing material.

The mathematical description of crack nucleation in a material with voids reduces to a two-dimensional problem of thermal elasticity for a perforated heat-releasing material with pre-fracture bands in the medium. The pre-fracture bands are aligned with the maximum tensile stresses.

Let there be a heat-releasing material with a biperiodic system of circular holes of radius λ ($\lambda < 1$) with the center at the points

$$P_{mn} = m\omega_1 + n\omega_2, \quad \omega_1 = 2, \quad \omega_2 = 2h e^{i\alpha}, \quad (1)$$

$$h > 0, \quad \text{Im } \omega_2 > 0 \quad (m, n = 0, \pm 1, \pm 2, \dots).$$

Symmetric straight-line pre-fracture bands emanate from the hole contours (Fig. 1). The contours of the circular holes [see (1)] are free from external loads. Heat transfer in fuel cells occurs mainly through heat conduction; therefore, the calculation of temperature fields reduces to the solution of problems of the heat-conduction theory in the presence of internal heat sources [2, 3]. Such calculations allow one to choose correctly the heat-release power in fuel cells and their basic dimensions, to find the value and determine the character of thermal stresses in the medium.

By virtue of symmetry of the boundary conditions and geometry of the domain D occupied by the material, the temperature and stresses are biperiodic functions with the fundamental periods ω_1 and ω_2 . The temperature $T(x, y)$ in the domain D is found by solving the heat-conduction equation

$$\Delta T + q/\delta = 0$$

with the condition on the hole contours

$$\frac{\partial T}{\partial r} = \frac{h_0}{\delta} (T_0 - T).$$

Here Δ is the Laplace operator, δ is the thermal conductivity of the medium, h_0 is the heat-transfer coefficient, and T_0 is the temperature of the cooling medium.

Interaction of the faces of the pre-fracture band (bonds between the faces) constrains crack nucleation.

In the mathematical description of interaction of the pre-fracture band faces, we assume that they have certain bonds with a prescribed deformation laws. The thermal loads on the material generate normal forces $p(x)$ in bonds connecting the pre-fracture band faces.

Hence, normal stresses $p(x)$ are applied to the band faces. These stresses are unknown in advance and have to be determined during solving the boundary-value problem of fracture mechanics with the conditions on the hole contours and pre-fracture band faces, respectively,

$$\sigma_r - i\tau_{r\theta} = 0, \quad \sigma_y - i\tau_{xy} = p(x). \quad (2)$$

The basic relations of the problem posed have to be supplemented by an expression relating the opening of the pre-fracture band faces and the forces in the bonds. Without loss of generality, we present this relation as [8]

$$v^+(x, 0) - v^-(x, 0) = C(x, p)p(x), \quad (3)$$

where the function $C(x, p(x))$ can be considered as tension-dependent effective compliance of bonds; $v^+ - v^-$ is the opening of the pre-fracture band faces.

For the critical heat-release intensity for crack nucleation to be determined, the problem formulation has to be supplemented by the condition (criterion) of crack emergence (breakdown of interparticle bonds in the material). We use the criterion of the critical opening of the pre-fracture band faces

$$v^+ - v^- = \delta_c, \quad (4)$$

where δ_c is the characteristic of material resistance to cracking.

The additional condition (4) allows one to determine the parameters of the heat-releasing medium at which the crack emerges.

Based on the Kolosov–Muskhelishvili formulas [9] and the boundary conditions on the contours of the circular holes and pre-fracture band faces (2), the problem reduces to determining two functions $\Phi(z)$ and $\Psi(z)$, which are analytical in the domain D , from the boundary conditions

$$\Phi_*(\tau) + \overline{\Phi_*(\tau)} - [\bar{\tau}\Phi'_*(\tau) + \Psi(\tau)] e^{2i\theta} + q_*\lambda^2/32 = 0; \quad (5)$$

$$\Phi_*(t) + \overline{\Phi_*(t)} + t\overline{\Phi'_*(t)} + \overline{\Psi(t)} + 3q_*t^2/32 = p(t), \quad (6)$$

where $\tau = \lambda e^{i\theta} + m\omega_1 + n\omega_2$ ($m, n = 0, \pm 1, \pm 2, \dots$), t is the affix of the points of the pre-fracture band faces, $q_* = \alpha E q / \delta$, α is the coefficient of linear temperature expansion, E is the modulus of elasticity of the material, $\Phi_*(z) = \Phi(z) - \alpha E F(z) / 4$, and $F(z)$ is a function analytical in the domain D , which satisfies the following boundary condition on the hole contours:

$$2 \operatorname{Re} \left[e^{i\theta} \left(F'(z) - \frac{q\bar{z}}{4\delta} \right) \right] = \frac{h_0}{\delta} \left(T_0 - F(z) - \overline{F(z)} + \frac{qz\bar{z}}{4\delta} \right). \quad (7)$$

Solution of the Boundary-Value Problem. The solution of the boundary-value problem (5)–(7) is sought in the form

$$F(z) = \beta_0 z^2 + \beta_1 \nu(z) + a_0 + \sum_{k=0}^{\infty} a_{2k+2} \frac{\lambda^{2k+2} \gamma^{(2k)}(z)}{(2k+1)!}; \quad (8)$$

$$\Phi_*(z) = \Phi_1(z) + \Phi_2(z), \quad \Psi(z) = \Psi_1(z) + \Psi_2(z); \quad (9)$$

$$\Phi_1(z) = \frac{1}{2\pi} \int_L g(x) \zeta(x-z) dx + A, \quad (10)$$

$$\Psi_1(z) = \frac{1}{2\pi} \int_L [\zeta(x-z) + Q(x-z) - x\gamma(x-z)] g(x) dx + B;$$

$$\Phi_2(z) = \beta_0^* z^2 + \beta_1^* \nu(z) + \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{\lambda^{2k+2} \gamma^{(2k)}(z)}{(2k+1)!},$$

$$\Psi_2(z) = dz^2 - \beta_1^* \zeta_*(z) + \sum_{k=0}^{\infty} \beta_{2k+2} \frac{\lambda^{2k+2} \gamma^{(2k)}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} \alpha_{2k+2} \frac{\lambda^{2k+2} Q^{(2k+1)}(z)}{(2k+1)!}, \quad (11)$$

$$\beta_0^* = -\alpha E \beta_0 / 4, \quad \beta_1^* = -\alpha E \beta_1 / 4,$$

where $\gamma(z)$ is the elliptic Weierstrass function, $\zeta(z)$ is the Weierstrass zeta function, $Q(z)$ is a special meromorphic function [10], $\nu(z) = -\iint \gamma(z) dz$, $\zeta_*(z) = -\int Q(z) dz$, $g(x) = (2\mu / (1 + \nu_0)) \partial[v^+(x, 0) - v^-(x, 0)] / \partial x$ is the sought function, μ is the shear modulus of the material, $\nu_0 = 3 - 4\nu$, ν is Poisson's ratio of the material, and A and B are constants. In Eqs. (10), the integrals are taken over the segment $L = [-l, -\lambda] \cup [\lambda, l]$.

Let us give the dependences that have to be satisfied by the coefficients of Eqs. (8)–(11). The conditions of symmetry with respect to the coordinate axes yield the equalities

$$\operatorname{Im} a_{2k} = 0, \quad \operatorname{Im} \alpha_{2k} = 0, \quad \operatorname{Im} \beta_{2k} = 0, \quad k = 1, 2, \dots$$

Because of biperiodicity of the temperature and stress fields and by virtue of self-consistency of the problem and periodicity of the main vector of forces acting on the arc connecting two congruent points in the domain D , we have

$$\begin{aligned} \beta_0 &= \frac{q}{16\delta\pi i} (\delta_1\bar{\omega}_2 - \delta_2\bar{\omega}_1), & \beta_1 &= \frac{q}{8\delta\pi i} (\bar{\omega}_1\omega_2 - \bar{\omega}_2\omega_1), & d &= \frac{q^*}{64\pi i} (\gamma_2\bar{\omega}_1 - \gamma_1\bar{\omega}_2), \\ (A + \bar{A})\omega_k + \bar{B}\omega_k &= \delta_k a + \bar{\gamma}_k a + \delta_k(a + \bar{a}) + \beta_2\lambda^2\delta_k + \alpha_2\lambda^2(\delta_k + \gamma_k) - \bar{D}_k \quad (k = 1, 2), \\ a &= -\frac{1}{2\pi} \int_L tg(t) dt, \\ D_k &= -\frac{2}{3}\beta_0^*(\omega_k^2\bar{\omega}_k - \bar{\omega}_k^3) + \beta_1^*\left(c_k - \frac{3}{8}\bar{\delta}_k\bar{\omega}_k^2 + \frac{1}{4}\delta_k\omega_k\bar{\omega}_k - \frac{1}{24}\gamma_k\omega_k^2\right), \\ \delta_k &= 2\zeta(\omega_k/2), \quad \gamma_k = 2Q(\omega_k/2) - \bar{\omega}_k\gamma(\omega_k/2), \\ c_k &= 2\bar{\xi}(\omega_k/2) - 2\nu_*(\omega_k/2) + \bar{\omega}_k\nu(\omega_k/2) \quad (k = 1, 2), \\ \xi(z) &= \iint \nu(z) dz, \quad \nu_*(z) = \int \zeta_*(z) dz. \end{aligned} \tag{12}$$

The notation of the constants in the biperiodic system is the same as that in [10]. The constants A and B are found from system (12), and these constants are real. They are conveniently presented as a sum of two constants:

$$A = A_* + A_{**}, \quad B = B_* + B_{**}.$$

Here A_{**} and B_{**} depend only on the coefficients α_2 and β_2 , i.e., are found from system (12) under the assumption that $a = 0$.

We can readily verify that the general presentations (8)–(11) determine a class of symmetric problems with a biperiodic distribution of temperature and stresses. The unknown function $g(x)$ and the constants α_{2k} and β_{2k} should be determined from the boundary conditions (6), (7). By virtue of biperiodicity, the system of the boundary conditions (6) is replaced by one functional equation, for instance, on the contour $\tau = \lambda \exp(i\theta)$, and system (7) is replaced by the boundary condition on L .

To obtain equations for the coefficients α_{2k} and β_{2k} of the functions $\Phi_2(z)$ and $\Psi_2(z)$, we present the boundary condition (6) in the form

$$\Phi_2(\tau) + \overline{\Phi_2(\tau)} - [\bar{\tau}\Phi_2'(\tau) + \Psi_2(\tau)] e^{2i\theta} + q_*\lambda^2/32 = f_1(\theta) + if_2(\theta), \tag{13}$$

where

$$f_1(\theta) + if_2(\theta) = -\Phi_1(\tau) - \overline{\Phi_1(\tau)} + [\bar{\tau}\Phi_1'(\tau) + \Psi_1(\tau)] e^{2i\theta}. \tag{14}$$

We assume that the function $f_1(\theta) + if_2(\theta)$ is decomposed on the contour $|\tau| = \lambda$ into a Fourier series. By virtue of symmetry, this series has the form

$$\begin{aligned} f_1(\theta) + if_2(\theta) &= \sum_{k=-\infty}^{\infty} A_{2k} e^{2ki\theta}, \quad \text{Im } A_{2k} = 0, \\ A_{2k} &= \frac{1}{2\pi} \int_0^{2\pi} (f_1(\theta) + if_2(\theta)) e^{-2ki\theta} d\theta \quad (k = 0, \pm 1, \pm 2, \dots). \end{aligned} \tag{15}$$

Substituting here expression (14) with allowance for Eq. (10), changing the order of integration, and calculating the integrals with the use of the residue theory, we find

$$\begin{aligned} A_0 &= -A - \bar{A} - \frac{1}{2\pi} \int_L g(t)f_0(t) dt, & A_2 &= B - \frac{1}{2\pi} \int_L g(t)f_2(t) dt, \\ A_{2k} &= -\frac{1}{2\pi} \int_L g(t)f_{2k}(t) dt, & k &= -1, \pm 2, \dots, \end{aligned}$$

$$f_0(t) = 2\zeta(t), \quad f_2(t) = \lambda^2 \gamma'(t)/2 + t\gamma(t) - \zeta(t) - Q(t),$$

$$f_{2k}(t) = \frac{(2k-1)\lambda^{2k}}{(2k)!} \gamma^{(2k-1)}(t) + \frac{\lambda^{2k-2}}{(2k-2)!} [\gamma^{(2k-3)}(t) - Q^{(2k-2)}(t) + t\gamma^{(2k-2)}(t)], \quad k = 2, 3, \dots,$$

$$f_{-2k}(t) = -\frac{\lambda^{2k}}{(2k)!} \gamma^{(2k-1)}(t), \quad k = 1, 2, \dots$$

Replacing the functions $\Phi_2(\tau)$, $\overline{\Phi_2(\tau)}$, $\Phi_2'(\tau)$, and $\Psi_2(\tau)$ in the left side of the boundary condition (13) by their expansions into Laurent series in the neighborhood of the point $z = 0$, substituting the Fourier series (15) into the right side of (13), and comparing the coefficients at identical powers of $\exp(i\theta)$, we obtain two infinite linear systems of algebraic equations with respect to the coefficients α_{2k} and β_{2k} [10, 11]:

$$\alpha_{2j+2} = \sum_{k=0}^{\infty} a_{j,k} \alpha_{2k+2} + b_j \quad (j = 0, 1, \dots), \quad a_{j,k} = (2j+1)\gamma_{j,k} \lambda^{2j+2k+2},$$

$$\gamma_{0,0} = \frac{3}{8} g_2 \lambda^2 + K_2 + \frac{2\lambda^2 K_0 K_3}{1 - 2\lambda^2 K_1} + \sum_{i=1}^{\infty} \frac{(2i+1)g_{i+1}^2}{2^{4i+4}} \lambda^{4i+2},$$

$$\gamma_{0,k} = -\frac{(2k+2)\rho_{k+1}}{2^{2k+2}} + \frac{(2k+4)!g_{k+2}\lambda^2}{2!(2k+2)!2^{2k+4}} + \frac{2\lambda^2 K_3 g_{k+1}}{2^{2k+2}(1 - 2\lambda^2 K_1)} + \sum_{i=1}^{\infty} \frac{(2j+2i+1)!g_{j+1}g_{k+i+1}\lambda^{4i+2}}{(2k+1)!(2i)!2^{2k+4i+4}} \quad (k = 1, 2, \dots),$$

$$\gamma_{j,0} = -\frac{(2j+2)\rho_{j+1}}{2^{2j+2}} + \frac{(2j+4)!g_{j+2}\lambda^2}{2!(2j+2)!2^{2j+4}} + \frac{2\lambda^2 K_0 g_{j+1}}{(1 - 2\lambda^2 K_1)2^{2j+2}} + \sum_{i=1}^{\infty} \frac{(2j+2i+1)!g_{i+1}g_{j+i+1}\lambda^{4i+2}}{(2j+1)!(2i)!2^{2j+4i+4}} \quad (j = 1, 2, \dots),$$

$$\gamma_{j,k} = \gamma_{k,j} = -\frac{(2j+2k+2)!g_{j+k+1}}{(2j+1)!(2k+1)!2^{2j+2k+2}} + \frac{(2j+2k+4)!g_{j+k+2}\lambda^2}{(2j+2)!(2k+2)!2^{2j+2k+4}} \quad (16)$$

$$+ \frac{1 + 2\lambda^2 K_1}{1 - 2\lambda^2 K_1} \frac{g_{j+1}g_{k+1}\lambda^2}{2^{2j+2k+4}} + \sum_{i=1}^{\infty} \frac{(2j+2i+1)!(2k+2i+1)!g_{j+i+1}g_{k+i+1}\lambda^{4i+2}}{(2j+1)!(2k+1)!(2i)!2^{2j+2k+4i+4}} \quad (j, k = 1, 2, \dots),$$

$$b_0 = A_2' - \frac{A_0' \lambda^2 K_3}{1 - 2\lambda^2 K_1} - \sum_{k=0}^{\infty} \frac{g_{k+2} \lambda^{2k+4}}{2^{2k+4}} A_{-2k-2}',$$

$$b_j = A_{2j+2}' - \frac{(2j+1)A_0' g_{j+1} \lambda^{2j+2}}{2^{2j+2}(1 - 2\lambda^2 K_1)} - \sum_{k=0}^{\infty} \frac{(2j+2k+3)!g_{j+k+2} \lambda^{2k+2j+4}}{(2j)!(2k+3)!2^{2j+2k+4}} A_{-2k-2}' \quad (j = 1, 2, \dots);$$

$$\beta_2 = \frac{1}{1 - 2\lambda^2 K_1} \left(2\lambda^2 K_0 - A_0' + 2 \sum_{k=1}^{\infty} \frac{g_{k+1} \lambda^{2k+2}}{2^{2k+2}} \alpha_{2k+2} \right),$$

(17)

$$\beta_{2j+4} = (2j+3)\alpha_{2j+2} + \sum_{k=0}^{\infty} \frac{(2j+2k+3)!g_{j+k+2} \lambda^{2j+2k+4}}{(2j+2)!(2k+1)!2^{2j+2k+4}} \alpha_{2k+2} - A_{-2j-2}' \quad (j = 0, 1, \dots).$$

Here

$$A_0' = \beta_1^* (1 - 2 \ln \lambda) - \frac{q_* \lambda^2}{32} - 2A_* - \frac{1}{2\pi} \int_L g(t) f_0(t) dt,$$

$$A_2' = \beta_0^* \lambda^2 + B_* - \frac{1}{2\pi} \int_L g(t) f_2(t) dt, \quad A_{-2}' = \beta_0^* \lambda^2 + A_{-2},$$

$$\begin{aligned}
A'_4 &= d\lambda^2 - \frac{3g_2\lambda^4}{64}\beta_1^* + A_4, & A'_{-2k-2} &= \frac{g_{k+1}\lambda^{2k+2}}{(2k+2)!2^{2k+2}}\beta_1^* + A_{-2k-2}, \\
A'_{2k+2} &= \beta_1^* \left(\frac{\lambda^{2k}\rho_k}{2^{2k}} - \frac{(2k+1)g_{k+1}\lambda^{2k+2}}{2^{2k+2}(2k+2)} \right) + A_{2k+2}, \\
g_k &= \sum'_{m,n} \frac{1}{T_0^{2k}}, & \rho_k &= \sum'_{m,n} \frac{\bar{T}_0}{T_0^{2k+1}}, & T_0 &= \frac{1}{2}P_{mn}, \\
K_0 &= \frac{\delta_1}{\omega_1} + \frac{2\pi i}{\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2}, & K_1 &= \frac{\pi i}{\omega_2\bar{\omega}_1 - \omega_1\bar{\omega}_2}, & K_3 &= K_0, \\
K_4 &= \frac{\gamma_1 - \delta_1}{\omega_1} - \frac{4\pi i}{\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2};
\end{aligned}$$

the prime at the summation sign means that the subscripts $m = n = 0$ do not participate in the summation process.

We require that functions (9)–(11) satisfy the boundary condition on the crack faces L and obtain a singular integral equation with respect to $g(x)$

$$\frac{1}{2\pi} \int_L g(t)K(t-x) dt + H(x) = p(x) \quad \text{on } L, \tag{18}$$

where

$$\begin{aligned}
K(x) &= 3\zeta(x) + Q(x) - x\gamma(x), & H(x) &= 3q_*\lambda^2/32 + 2A + B + 2\Phi_2(x) + x\Phi_2'(x) + \Psi_2(x), \\
2A + B &= [(a + \alpha_2\lambda^2)(\delta_1 + \gamma_1) + (2a + \beta_2\lambda^2)\delta_1 - D_1]/\omega_1.
\end{aligned}$$

The singular integral equation (18) and also systems (16) and (17) are the governing resolving equations of the problem, which allow determining the function $g(x)$ and the coefficients α_{2k} and β_{2k} . Knowing the functions $\Phi_2(z)$, $\Psi_2(z)$, and $g(x)$, we can find the stress-strain state of the heat-releasing material with pre-fracture bands.

Technique of the Numerical Solution and Analysis. Using, in the basic parallelogram of the periods, the expansions

$$\begin{aligned}
\zeta(z) &= \frac{1}{z} - \sum_{j=1}^{\infty} \frac{g_{j+1}z^{2j+1}}{2^{2j+2}}, & \gamma(z) &= \frac{1}{z^2} + \sum_{j=1}^{\infty} \frac{(2j+1)g_{j+1}z^{2j}}{2^{2j+2}}, \\
Q(z) &= \sum_{j=1}^{\infty} \frac{(2j+2)\rho_{j+1}}{2^{2j+2}} z^{2j+1},
\end{aligned}$$

we can bring Eq. (18) to the conventional form

$$\frac{1}{\pi} \int_L \frac{g(t) dt}{t-x} + \frac{1}{\pi} \int_L g(t)K_0(t-x) dt + H(x) = p(x). \tag{19}$$

Replacing the variables as $t = \xi l$ and $x = \xi_0 l$ and applying some simple transformations, we bring Eq. (19) to the form

$$\begin{aligned}
\frac{1}{\pi} \int_{L_0} \frac{g_*(\xi)}{\xi - \xi_0} d\xi + \frac{1}{\pi} \int_{L_0} g_*(\xi)K_0(\xi - \xi_0) d\xi + H_*(\xi_0) &= f_*(\xi_0), \\
g_*(\xi) &= g(t), & L_0 &= [-1, -\lambda_1] \cup [\lambda_1, 1], & \lambda_1 &= \lambda/l, \\
K(\xi) &= \sum_{j=0}^{\infty} K_j \left(\frac{l}{2}\right)^{2j+2} \xi^{2j+1}, & K_0 &= \omega_1 \operatorname{Re} \delta_1, & K_j &= g_{j+1}, \\
K_0^* &= -\frac{\omega_1}{2}(\bar{\gamma}_1 + \bar{\delta}_1), & K_*(\xi) &= \sum_{j=0}^{\infty} K_j^* \left(\frac{l}{2}\right)^{2j+2} \xi^{2j+1},
\end{aligned} \tag{20}$$

$$K_j^* = (j+1)(\rho_{j+1} - g_{j+1}), \quad j = 1, 2, \dots,$$

$$K_0(\xi) = K_*(\xi) - K(\xi), \quad K_*(\xi_0) = H(\xi_0 l), \quad f_*(\xi_0) = p(\xi_0 l).$$

Taking into account that $g_*(\xi) = -g_*(-\xi)$, we can write Eq. (20) as

$$\frac{2}{\pi} \int_{\lambda_1}^1 \frac{\xi g_*(\xi)}{\xi^2 - \xi_0^2} d\xi + \frac{1}{\pi} \int_{\lambda_1}^1 K_0^*(\xi, \xi_0) g_*(\xi) d\xi + H_*(\xi_0) = f_*(\xi_0),$$

$$K_0^*(\xi, \xi_0) = K_0(\xi - \xi_0) + K_0(\xi + \xi_0), \quad \lambda_1 \leq \xi_0 \leq 1. \quad (21)$$

We transform Eq. (21) to a form more convenient for finding its approximate solution by replacing the variables

$$\xi^2 = u = \frac{1 - \lambda_1^2}{2} (\tau + 1) + \lambda_1^2, \quad \xi_0^2 = u_0 = \frac{1 - \lambda_1^2}{2} (\eta + 1) + \lambda_1^2.$$

Thereby, the interval of integration $[\lambda_1, 1]$ transforms to the interval $[-1, 1]$, and the transformed Eq. (21) acquires the standard form:

$$\frac{1}{\pi} \int_{-1}^1 \frac{g_*(\tau)}{\tau - \eta} d\tau + \frac{1}{\pi} \int_{-1}^1 g_*(\tau) B(\eta, \tau) d\tau + H_*(\eta) = f_*(\eta). \quad (22)$$

Here

$$g_*(\tau) = g_*(\xi), \quad B(\eta, \tau) = \frac{1 - \lambda_1^2}{2} \sum_{j=0}^{\infty} (K_j^* - K_j) \left(\frac{l}{2}\right)^{2j+2} u_0^j A_j,$$

$$A_j = \left[2j + 1 + \frac{(2j+1)(2j)(2j-1)}{1 \cdot 2 \cdot 3} \left(\frac{u}{u_0}\right) + \dots + \frac{(2j+1)(2j)(2j-1) \dots 1}{1 \cdot 2 \cdot 3 \dots (2j+1)} \left(\frac{u}{u_0}\right)^j \right],$$

$$H_*(\eta) = H_*(\xi_0), \quad f_*(\eta) = f_*(\xi_0).$$

To construct the solution of the singular integral equation, we use the method of solving singular integral equations described in [12, 13]. In addition to the singularity in the Cauchy kernel, the singular integral equation (22) has a motionless singularity at the point where the pre-fracture band reaches the surface of the circular hole. In this case, the function $g(x)$ at the points $x = \pm\lambda$ has a singularity, which differs from the root singularity. The character of this singularity can be determined by analyzing the integral equation (22) [14]. In contrast to the case

with the internal pre-fracture band, the integral is $\int_0^l g(t) dt = C \neq 0$. The constant C is expressed through the opening of the pre-fracture band on the surface of the circular hole and has to be determined after the singular integral equation is solved.

In the case considered, we should have used the method of solving the integral equation based on the Gauss–Jacobi quadrature formula. As the expressions for the functions $B(\eta, \tau)$ and $H_*(\eta)$ are extremely cumbersome, it is difficult to determine the singularities of the function $g_*(\eta)$ at the points $x = \pm\lambda$. In addition, it should be noted that the gain in convergence reached in the refined method is lost because the formulas for the matrix coefficients in the system are too cumbersome. We use another, simplified approach to the numerical solution of integral equations of the type (22). The efficiency of this method has been verified by solving numerous problems [4, 11–13, 15]. As the stresses in the heat-releasing material are bounded, the solution of the singular integral equation (22) has to be found in the class of functions bounded everywhere. Let us present this solution in the form

$$g_*(\eta) = g_0(\eta) \sqrt{1 - \eta^2},$$

where $g_0(\eta)$ is a new unknown regular function.

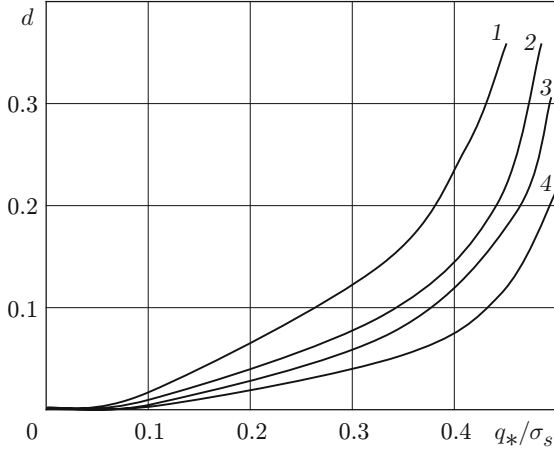


Fig. 2

Fig. 2. Dimensionless length of the pre-fracture band d versus the heat-release intensity q_*/σ_s for different values of the hole radius: $\lambda = 0.2$ (1), 0.3 (2), 0.4 (3), and 0.5 (4).

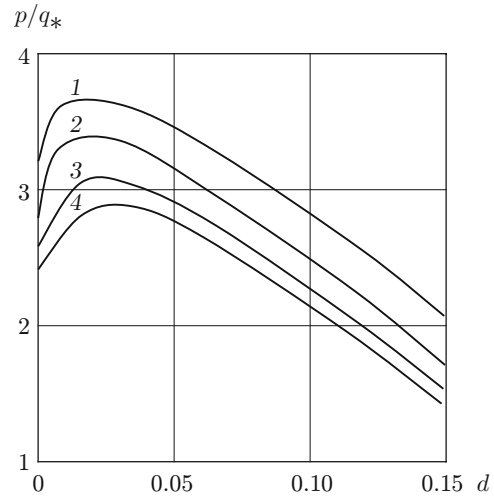


Fig. 3

Fig. 3. Forces in the bonds p/q_* versus the dimensionless length of the pre-fracture band (notation the same as in Fig. 2).

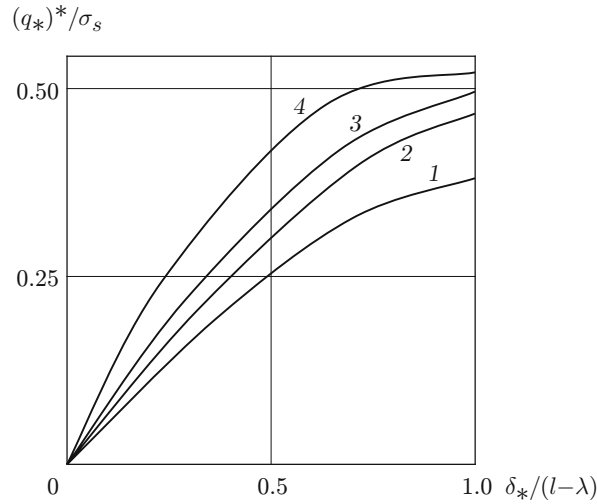


Fig. 4. Critical heat-release intensity in the material $(q_*)^*/\sigma_s$ versus the dimensionless opening $\delta_*/(l - \lambda)$ for $x = \pm\lambda$ (notation the same as in Fig. 2).

The first step of iterative calculations is solving system (16), (17), (23), (26) for linearly elastic interparticle bonds. Subsequent iterations are performed only if $v^+ - v^- > v_*$ in some part of the pre-fracture zone. In such iterations, one has to solve a system of equations in each approximation for quasi-elastic bonds with effective compliance varied along the faces of the pre-fracture band and depending on the magnitude of forces in the bonds, which is calculated at the previous step. The calculation of effective compliance is similar to determining the secant modulus in the method of variable parameters of elasticity [17]. The process of consecutive approximations is terminated when the forces along the pre-fracture band at two successive iterations are almost identical.

The nonlinear part of the curve of bond deformation was approximated by a bilinear dependence [8] whose ascending segment corresponded to deformation of bonds ($0 < v^+ - v^- \leq v_*$) with the maximum tension. At $v^+ - v^- > v_*$, the deformation law was a nonlinear dependence determined by the points (v_*, σ_*) and (δ_c, σ_c) , with an increasing linear dependence at $\sigma_c \geq \sigma_*$ (linear hardening corresponding to elastoplastic deformation of bonds).

To determine the critical equilibrium state of the pre-fracture band at which the crack appears, we use condition (4). From the solution obtained, we calculate the displacement $v(x, 0)$ on the pre-fracture band:

$$v(x, 0) = -\frac{1 + \varkappa_0}{2\mu} \int_{-l}^x g(x) dx.$$

At $x = \lambda$, the distance between the faces of the pre-fracture band is

$$v(-\lambda, 0) = -\frac{1 + \varkappa_0}{2\mu} \int_{-l}^{-\lambda} g(x) dx.$$

Thus, the condition determining the critical intensity of internal heat sources in the material (loading parameter q_*) at which a crack is formed at the point $x = \pm\lambda$ is

$$C(\lambda, p(\lambda))p(\lambda) = \delta_c. \quad (27)$$

The solution of the algebraic system (16), (17), (23), (26), (27) allows determining the critical intensity of internal heat sources, the size of the pre-fracture band, and the forces in the bonds in the state of critical equilibrium at which cracks start forming in the material.

Figure 2 shows the dimensionless length of the pre-fracture band $d = (l - \lambda)/\lambda$ as a function of the dimensionless heat-release intensity q_*/σ_s for a square grid of holes (σ_s is the tensile yield point of the material).

Figure 3 shows the forces in the bonds p/q_* versus the dimensionless size d for a square grid of holes.

Figure 4 shows the critical heat-release intensity in the material $(q_*)^*/\sigma_s$ as a function of the dimensionless opening $\delta_*/(l - \lambda)$ at the point $x = \lambda$ for a triangular grid of holes ($\delta_* = \pi\delta_c\mu/[(1 + \varkappa_0)\sigma_s]$).

An analysis of the critical equilibrium state of the perforated heat-releasing material at which the crack appears reduces to a parametric study of the resolving algebraic system (16), (17), (23), (26) and the criterion of crack emergence (27) with different laws of bond deformation, thermophysical and elastic constants of the material, and geometric characteristics of the latter. The forces in the bonds and the opening of the pre-fracture band are found directly by solving the resultant algebraic systems in each approximation.

REFERENCES

1. S. L. Sobolev and G. V. Mukhina, "Determining thermal stresses in a medium containing voids," *Atom. Énerg.*, **5**, No. 2, 178–181 (1958).
2. I. I. Gol'denblat and N. A. Nikolaenko, *Calculation of Temperature Stresses in Nuclear Reactors* [in Russian], Gosatomizdat, Moscow (1962).
3. L. A. Fil'shtinskii, "Problems of heat conduction and thermal elasticity for a plate attenuated by a biperiodic system of identical circular holes," in: *Thermal Stresses in Structural Elements* (collected scientific papers) [in Russian], No. 4, Naukova Dumka, Kiev (1964), pp. 103–112.
4. V. V. Panasyuk, *Mechanics of Quasi-Brittle Fracture of Materials* [in Russian], Nauk. Dumka, Kiev (1991).
5. B. N. Cox and D. B. Marshall, "Concepts for bridged cracks fracture and fatigue," *Acta Metall. Material*, **42**, No. 2, 341–363 (1994).
6. R. V. Goldstein and M. N. Perelmuter, "Modeling of bonding at the interface crack," *Int. J. Fracture*, **99**, Nos. 1/2, 53–79 (1999).
7. R. V. Goldstein and M. N. Perelmuter, "Crack at the interface of materials with bonds between the crack faces," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 1, 94–112 (2001).
8. R. V. Goldstein and M. N. Perelmuter, "Growth of cracks at the interface of materials," in: *Problems in Mechanics* (collected scientific papers devoted to the 90th anniversary of A. Yu. Ishlinskii) [in Russian], Fizmatlit, Moscow (2003), pp. 221–238.

9. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen (1963).
10. E. I. Grigolyuk and L. A. Fil'shtinskii, *Perforated Plates and Shells* [in Russian], Nauka, Moscow (1970).
11. V. M. Mirsalimov, *Fracture of Elastic and Elastoplastic Solids with Cracks* [in Russian], Elm, Baku (1984).
12. V. V. Panasyuk, M. P. Savruk, and A. P. Datsyshin, *Stress Distributions in the Vicinity of Cracks in Plates and Shells* [in Russian], Naukova Dumka, Kiev (1976).
13. V. M. Mirsalimov, *Non-One-Dimensional Elastoplastic Problems* [in Russian], Nauka, Moscow (1987).
14. N. I. Muskhelishvili, *Singular Integral Equations* [in Russian], Nauka, Moscow (1968).
15. M. P. Savruk, P. N. Osiv, and I. V. Prokopchuk, *Numerical Analysis in Plane Problems of the Crack Theory* [in Russian], Naukova Dumka, Kiev (1989).
16. A. A. Il'yushin, *Plasticity* [in Russian], Gostekhteorizdat, Moscow–Leningrad (1948).
17. I. A. Birger, “General algorithms of solving problems of the elasticity, plasticity, and creep theories,” in: *Achievements of Mechanics of Deformable Media* [in Russian], Nauka, Moscow (1975), pp. 51–73.